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The Possibility of Moving to The Tree games

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Abstract— This research presents the Domineering games and study the possibility of moving from it to the tree games, with the help of several mathematical notions.

Index Terms—Game theory, Gameswithout Chance, Combinatorial games, The Tree games, Young tableau, Hook length, The Domineering games.

1 Introduction

In1902, combinatorial game theory was born by Bouton at Harvard[1]. In the 1930s, Sprague and Grundy extended this theory to cover all impartial games. In the 1970s, the theory was extended to partisan games, a large collection which includes the ancient Hawaiian game called Konane [2], many variations of Hackenbush, cutcakes, ski-jumps, Domineering, Toads-and-Frogs, etc [3]. One reviewer remarked that although there were over a hundred such games in Winning Ways, most of them had been invented by the authors. John Conway axiomatized this important branch of the subject [4].

The branch of mathematics known as Combinatorial Game Theory may be widely known by name, but actual knowledge of the subject is not as common. Thus, before we can really get into the main subject of Trees, we need to lay the groundwork for Combinatorial Game Theory itself. Game Theory is defined as “the branch of mathematics concerned with the analysis of strategies for dealing with competitive situations where the outcome of participant’s choice of action depends critically on the actions of other participants” [5].

Combinatorial Game Theory has several important features that sets it apart [6,7]. Primarily, these are games of pure strategy with no random elements. Specifically:
1. There are Two Players who Alternate Moves;
2. There are No Chance Devices hence no dice or shuffling of cards;
3. There is Perfect Information all possible moves are known to both players and, if needed, the whole history of the game as well.
4. Play Ends, regardless even if the players do not alternate moves, the game must reach a conclusion;
5. A player that is unable to move loses.
2 Procedure for Paper Submission

1.1 Game Rules and Game abbreviations

We will try to explain how to deal with the Games and we'll show the rules.

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1.1.1 Various Game Values

If we have a game $G$and this game and This game include options for player L named G_L and options for player R namedG_R [8]:

- $G = \{G_L \quad | \quad G_R\}$
- $-G = \{-G_L \quad | \quad -G_R\}$
- $G = H \quad \text{if } G - H = 0$
- $0 = \{\ |\ \}$

If G = 0, then the first player to move loses

- $1 = \{0\ |\ \}$

A positive game value is a Left win

- $-1 = \{|0\ \}$

A negative game value is a Right win
From the above, we find:

- $1 = \{0\ |\ \}.\, 2 = \{1\ |\ \}\,.\, 3 = \{2\ |\ \}$
- $-1 = \{\ |\ 0\}.\, -2 = \{\ |\ -1\}\,.\, -3 = \{\ |\ -2\ \}$
- $n = \{n-1\ |\ \}$
- $* = \{0\ |0\ \}$
- $* = -*$
- $*n \ = \{*0 .*1. \dots .*n-1 \mid *0 .*1. \dots .*n-1\ \}$
- $\uparrow = \{0\ |\ *\}$
- $\downarrow = -\uparrow$
- $G > 0\ means\ Left\ wins$
- $G < 0\ means\ Right\ wins$
- $G = 0$ means first player loses
- $G \| 0$ means second player loses

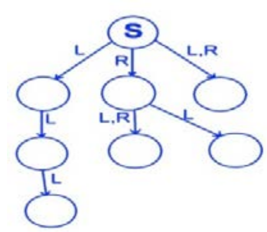

Fig. 1. sample game with two players, left and right, alternate moves

If you have no move, you lose and your opponent wins and games must terminate in finite time.

1.1.2 Definition

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A game is an ordered pain of sets of games and we write a game as {left's moves | right's moves Or {left'soptions | right's options } Or
$G = \{G_L \mid G_R\}$. For example, the empty set of games, so{ | } = 0 is a game, won by the second player. We now have two sets of games:{ }and{0 }. so we can form games{0 | } = 1 left wins {|0 } = −1 right wins{0 |0 } =∗ first player wins We now have 16 set of {0 .1. −1.∗ } and we can write:

Games	Born on day $n =$
{ | } = 0	0
{0 | } = 1 {|0 } = −1 {0 |0 } =∗	1
e.g. {0 .∗ | − 1 }. {1 .∗ |0.1 }	2
etc.	etc.

2. DOMINEERINGGAMES

We must first briefly discuss a couple of topics, the first of which is another game known as Domineering. The premise of the game is simple; each player takes turns placing a domino, made of two tiles, on a tiled game board (similar to a chess board), with Left placing their piece vertically and Right placing their piece horizontally. Pieces are not allowed to overlap

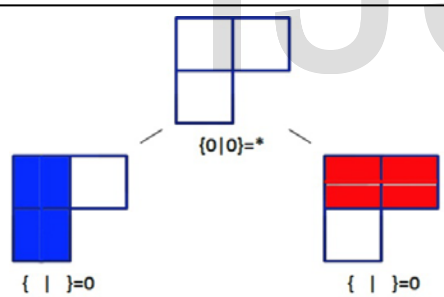

Fig. 2. A sample game of Domineering with moves included

and cannot be played outside the boundaries of the board. The first one that is unable to play loses [9]. As an example, observe the following game where blue is Left's vertical move and red is Right's horizontal move:
One problem with looking at Domineering in this form is it is harder to visually distinguish the subgame. Therefore, when drawing game trees of Domineering, whenever a domino is places onto the board, the covered cells are removed from the drawing [10]. See figure 3 for an example.

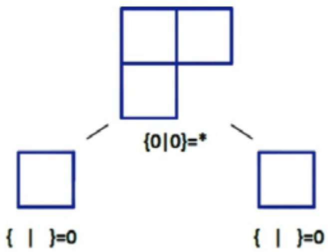

Fig. 3. A sample game of Domineering eliminating moves.

In that form, we can more easily see that the two games resulting from Left's and Right's individual moves actually result in the same game, thus having the same value.
Now,there is one concept that needs to be addressedwithDomineering. Whereas with all our Hackenbush games, we had $a \le b$ for all games of value$\{a|b\}$, it is possible for games in

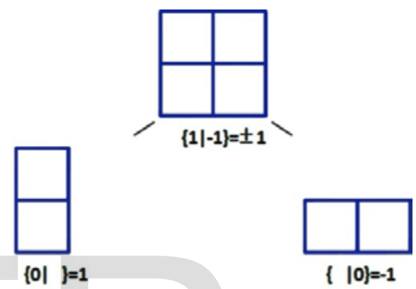

Fig. 4. A sample game of Domineering eliminating moves.

Domineering to play out such that $a > b$. as follows:
this game was assigned the value ±1. This is an example of a switch game, as the value can switch depending on who makes the first move. In this example, the first player to move gains a 1-move advantage. The way this breaks down is as follows: For a game $\{y|z\}$ such that $y > z$

$\{y|z\} = a + \{x|-x\} = a \pm x$where $a = \frac{y+z}{2}$ and $x = \frac{y-z}{2}$

Now, keeping in mind that, for Domineering, negating the value involves turning
the board by 90^o,we can look at an example that is a bit more

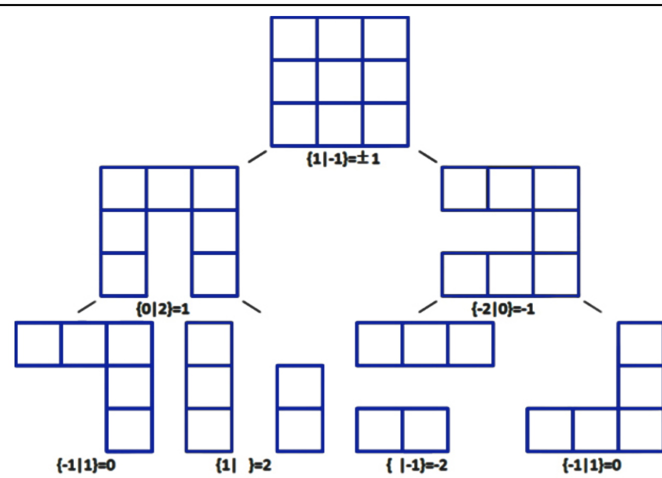

Fig. 5. Domineering played on a 3x3 board

complicated.

2.1. Comparison between Combinatorial Games in general and Domineering Game

Now we will compare between Domineering and Hackenbush Games, depending on many important points

Combinatorial Games in general	Domineering Game
$a \leq b$ for all games of value $\{a\|b\}$	Maybe not like this game $\{1\|-1\} = \pm 1$
We find for every game "G" there is – Gwhere $\mathrm{G} - \mathrm{G} = 0$.	The same we find for every game "G" there is – Gwhere $\mathrm{G} - \mathrm{G} = 0$. But to get the negation of the game, you should rotate 90^o to the right direction
There is one value of the game	Maybe not
There are up and down games	There are no games like this

2.2. Young Tableau and Hook Length

Our second set of topics before getting into Trees are the Young tableau and hook length. A Young tableau can be formed by taking a partition of a positive integer and filling out a tableau of squares with left justifiedrows of length equal to the partitions in decreasing order [11,12].

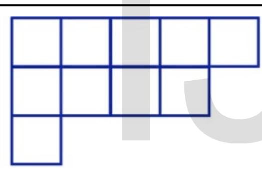

Fig. 6. A partition of (5,4,1) would have 5 squares on the top row, then 4 in the middle, and 1 on the bottom row, all alignedon the left side.

Astandard Young tableau of a partition of n has distinct integers from 1 to n such that each row and column form increasing sequences.

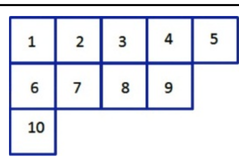

Fig. 7. A standard Young tableau of partition (5,4,1).

Related to the Young tableau is the concept of the hook and its hook length. Let a Young tableau have a shape denoted by λ. A hook, $H\lambda(i.j)$ on a Young tableau is the subset of cells on the tableau starts at the $(i.j)$ and continues right and down from there until the column and row terminate [11,13]. The hook length of $H\lambda(i.j)$. denoted $\mathrm{h}\lambda(\mathrm{i.j})$. is the total number of cells in $H\lambda(i\,.\mathrm{j})$

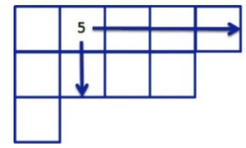

Fig. 8. A visual representation of $\mathrm{H}\lambda(1.2)$ filled with its hook length of 5.

The number of standard Young tableaus of a shape λ. denoted $d\lambda$ can be calculated.

By$d_\lambda = \frac{n!}{\Pi h_\lambda(i.j)}$.

The easiest was to obtain $\Pi h_\lambda(i.j)$ would be to fill out the tableau with all the corresponding hook lengths, like in figure 9.

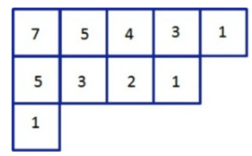

Fig. 9. The same tableau filled with the hook lengths of each cell.

Using this as an example, we can see that for a tableau of shape$\lambda = (5.4.1)$ we get

$d_\lambda = \frac{10!}{7.5.5.4.3.3.2.1.1.1} = 288.$

This concept has since been generalized to binary trees to the effect of the equation not changing at all.

2.2.1. Example:

in figure 10,11 we'll study (4,2,1) game, $\lambda = (4.2.1)$

we get $d_\lambda = \frac{7!}{6.4.3.2.1.1.1} = 35$

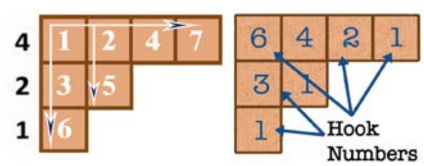

Fig. 10. A standard Young tableau of partition (4,2,1) filled with the hook numbers.

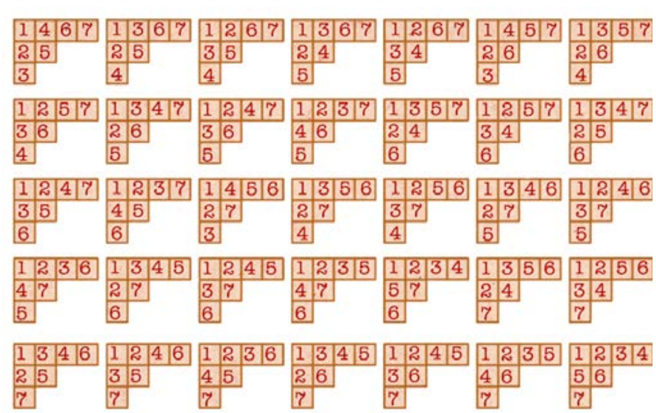

Fig. 11. (35 cases of (4,2,1) game)

2.3 The Possibility of Moving to the Trees

The inspiration for Trees came from the concept presented in the previous section where hook length for a Young tableau was generalized for binary tree. Using Domineering as a basis, Trees began with the premise of looking at Domineering as a rooted binary tree suchthat every square is a vertex and every vertex of adjacent squares are joined by an edge,with a vertical connection slanting from right to left and a horizontal connection slanting from left to right [14].

2.3.1 Example:

in figure.12 a Domineering game is moved to a tree game

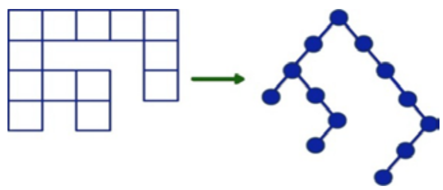

Fig. 12. A Domineering game with its equivalent Tree game.

However, while at first glance it seems as though Trees is nothing but a restricted version of Domineering, there are in fact games that are exclusive to each particular game (See figure 13).

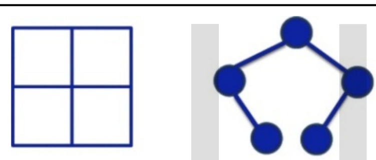

Fig. 13. Examples of games exclusive to Domineering and Trees respect-tively.

For the Domineering game in figure.13, if we were to convert it into a graph, we would end up with a cycle graph C4. As for the Trees game, it cannot directly be converted to a tableau since the bottom two vertices would cause the corresponding cells to overlap.

Before moving on, we need to clarify some terms that will be used intermittently**.** For starters, the premise of Trees plays very similar to Domineering. Converting from Domineering to Trees, we can define our analog of what a piece is [15].

2.3.2. Definition:

A piecein Trees refers to two vertices joined by one edge Furthermore, the following definitions give name to certain parts of a Trees game board.

2.3.3. Definition:

A contested vertexin this game refers to a vertex that is in piecesthat may be taken by either player.

2.3.4. Definition:

A free vertexin this game refers to a vertex that is in pieces that
may only be taken by a specific player.

2.3.5. Definition:

We call a piece an**extension**of another piece if the two pieces share a vertex and both pieces belong to the same player.

2.3.6. Definition:

We call a set of pieces a leg if the following conditions are met:

- Every piece in the set shares at least one vertex with another piece in the set.
-Exactly one vertex in the set is a contested vertex.
-Exactly one vertex in the set is a leaf.
- The maximum degree of all free vertices in the set is 2.

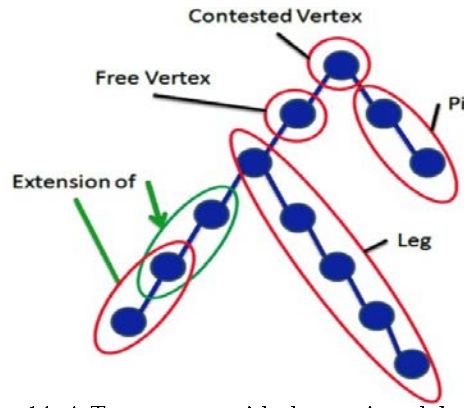

Fig. 14. A Trees game with the sections labelled**.**

2.4 Gameplay

Gameplay follows by players alternatingly taking their pieces from the board, with Left taking pieces that slant from right to left and Right taking the opposite. The first player unable to remove a piece loses [16]. Take the following simple game tree for example. Note that, when a piece is removed, all edges connecting to that piece colored green in following figures, are suddenly useless, and therefore are removed from successive plays. For example:

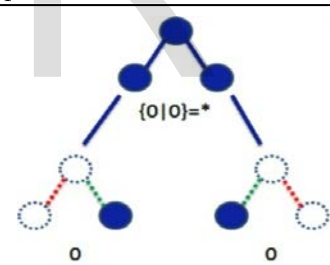

Fig. 15. Example of Trees game, marking removed pieces for clarity.

As with Domineering, various games from Trees can be a switch game
as well, such as the following:

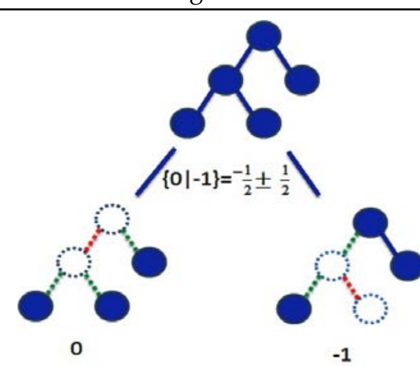

Fig. 16. A switch game in Trees.

Other properties used in finding the value of this example include negating the game by performing a horizontal flip on the entire tree [17,12]. However, while a vertical flip can also have the same effect, it violates the general structure of the game as the root would then be at the bottom. Many different values can be found just through experimentation alone:

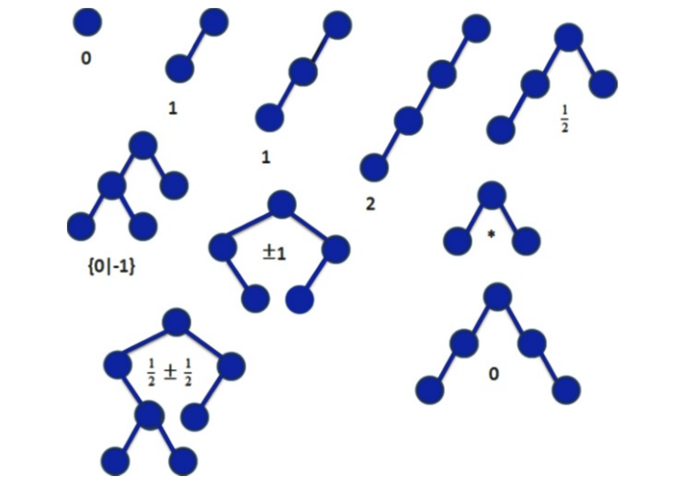

Fig. 17. Various Trees games and their values.

Next, we have a theorem that can help quickly find values of games by adding to the shape in certain ways.

2.4.1. Theorem:

Let G be a game with at least one leg for Left. Then, adding two extensions to that leg increases the value of the game by 1.

Let G be a game and M_n be a move for either player. Then $G - M_n = z$ refers the game Ghaving the move M_n removed from it and resulting in a game with a value of$z$.

Proof. Let G be a game such that$G = \{a|b\}$ and G has at least one leg for Left with terminal vertex v. Let G'be a game resembling G except with two extensions extended from v, adding vertices w and x. Let Right's best move in G be M_R. Then$G - M_R = b$. Then, Right's best move in G' is M_R.

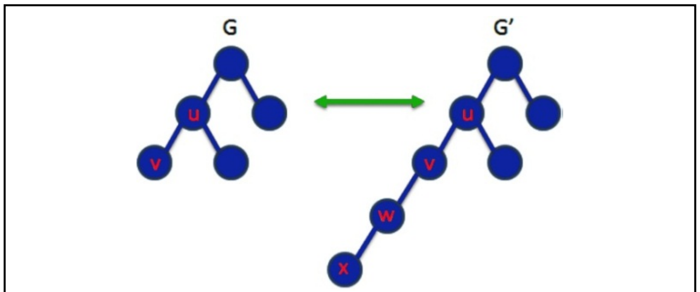

Fig. 18. A general game for visualization purposes.

Case 1. Assume M_R includes a contested vertex u such that the path from u to v is a leg, called L. WLOG, $let\ |L| = 1$. Then $G' - M_R$still contains the path wx. Thus
$G' - M_R = b + 1$.

Case 2. AssumeM_R does not include the contested vertex in L. Then the proof is trivial and $G' - M_R = b + 1$.

Now let Left's best move in G be M_L. Then $G - M_L$ = a. We want to show that, if M'_L is Left's best move in G' , then $M'_L = M_L$. that is, adding the two extensions does not change Left's best move.

Assume M'_L L lies on the extension, that is M'_L removesthe piece vw or wx. However, since u is a contested vertex,then apiece including u would be preferable. Thus M_0L cannot exist on the extension. That means $M'_L = M_L$.

Case 1. Let $M_L \subset$ L. Then M_L removes the piece uv. Let WX be the game consisting only of the piece wx. Then $G' - M_L = G - M_L + wx = a + 1$.

Case 2. Let $M_L \subset$ L. Then the proof is trivial and $G' - L = a + 1$. Thus, in general. $G' = \{a + 1|b + 1\} = \{a|b\} + 1$.

2.4.2. Remark:

It should be noted that there are caseswhereone extension can increase the value of a game by 1, but that does not always occur. Two extensions, however, will always increase the game value by 1. For example, see figure 19.

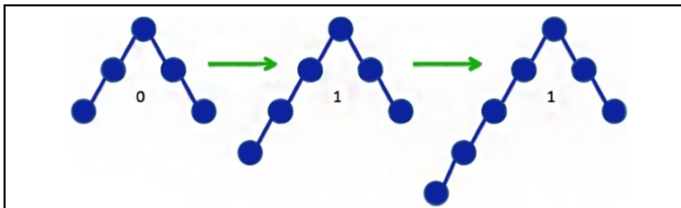

Fig. 19. A Trees game showing how the value is affected by one and two extensions.

Naturally, this theorem can be reworked to apply to Right's moves as well.

2.4.3. Corollary:

Let G be a game with at least one leg for Right. Then, adding two extensions to that leg decreases the value of the game by 1.

Proof. Since performing a horizontal flip on a Trees game negates the value, the proof is trivial.

2.4.4. Remark:

Since Trees and Domineering have similar structures, this theorem and corollary can quickly be applied to Domineering as well by adding a set of two horizontal or vertical squares.

One of the most basic structures of a Trees game is a game with two legs of equal length meeting at one contested vertex. The following theorem will show that there are only two possible values for games of this particular shape.

2.4.5. Example:

Let $G(n)$ be a game with exactly one contested vertex and all pieces form two legs of equal length n. Then we will try to discuss the following two situations:

(a) If n is odd, then what's the result?
(b) If n is even, then what's the result?

To begin, we know $G(0) = 0$ and $G(1) = *$. Let a game $G(n) = \{a|b\}$. Let $m \in N$. By Theorem2.4.1, adding two extensions to Left increases the value by 1 and by Corollary2.4.3, adding two extensions to Right decreases the value by 1. Since there are an equal number of extensions on either side, the increase/decrease in value is nullified.

(a) If n is even. $G(n) = G(0 + 2m) = G(0) = 0$.
(b) If n is odd. $G(n) = G(1 + 2m) = G(1) = *$.

This is as obvious as the figure 20.

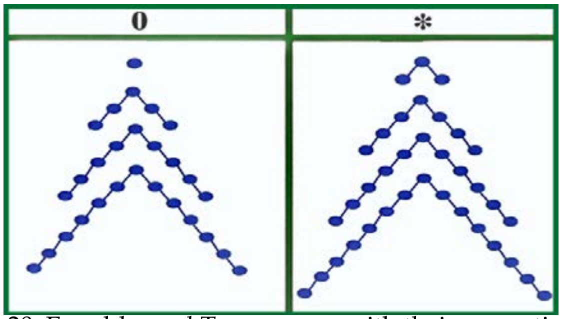

Fig. 20. Equal-legged Trees games with their respective values.

3. Conclusion

In this paper, we have explained one model of combinatorial games, and performed a simple comparison between Combinatorial Games in general and Domineering Game in particular. The key point in the research is the fact that we have studied the possibility of moving to the trees. Therein we explained how to increase the value of the game by the amount of (1). Most importantly, this was done after we clarified all the concepts related to the trees.

4. Future Potential Topics

After the moving from Domineering games to Trees games. Trees games have an easier structure to interpret, potentially making studies of these games easier. We moved to the field like The Graph, it would seem relevant to try to apply results from Graph to Domineering or another combinatorial game, like Nim game, where we can use many algebraic operations.

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